Extensions and Generalizations of Minimal Unsatisfiability

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Abstract

Motivated by practical applications, the concepts related to minimal unsatisfiability and minimal unsatisfiable subformulas (MUSes) have recently been extended beyond propositional formulas in CNF, and, also, generalized to the case of satisfiable formulas. In this talk I will discuss the algorithms and the optimization techniques for the computation of MUSes in some of these extended and generalized domains.

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Related areas

- Minimal Unsatisfiable sets of Constraints (MUCs) — in CSP.
- Irreducible Infeasible Subsets (IISes) in linear programs Note: many MUS extraction algorithms can be traced back here.

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- Note the special role of $\mathcal{G}_0$ — the "background" clauses. Example: in PBA a group = CNF representation of a latch, $\mathcal{G}_0$ is the rest of the circuit.
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Perhaps the *most* useful (from practical standpoint) and versatile generalization of MUSes.

Computing Group-oriented MUSes

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- Some low-level modification to a SAT solver are possible to optimize SAT solver use [Ryvchin and Strichman, SAT'11].

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- Very effective.

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- Effectiveness varies, depending on the structure of GCNFs.

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- Effectiveness is not clear.

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What is a subset-minimal set of variables sufficient to refute $\mathcal{F}$?

$\{p, q\}$ is such a subset of variables.

$\{p, q\}$ is a *variable-MUS* of $\mathcal{F}$.

Variable MUSes

Def: The subformula of CNF $\mathcal{F}$ *induced* by $V \subseteq Var(\mathcal{F})$ is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

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- The subformula induced by $\{p\}$, $\mathcal{F}|_{\{p\}} = \{C_1\}$.
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- Hence, $\{p, q\}$ is a VMUS of $\mathcal{F}$. Notation: $\{p, q\} \in \text{VMUS}(\mathcal{F})$.

Computing Variable MUSes

Basic algorithm is similar to deletion-based MUS extraction algorithm: based on detection of *necessary* variables.

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Properties:

1. $V \in \text{VMUS}(\mathcal{F})$ if and only if every $v \in V$ is necessary for $\mathcal{F}|_V$.
2. If v is necessary for $\mathcal{F}$, then v is necessary for any unsatisfiable $\mathcal{F}' \subseteq \mathcal{F}$.
3. Fact: v is necessary for $\mathcal{F}$ iff for some τ, $\tau(\mathcal{F} \setminus \mathcal{F}^v) = 1$, and $\tau(\mathcal{F}^v) = 0$.

Computing Variable MUSes: optimizations [Belov et al, SAT'12]

Model rotation — needs to be modified:

- When $\mathcal{F} \setminus \mathcal{F}^v \in \text{SAT}$, the model falsifies some clauses with v.

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- Fact: For any τ, any variable shared among clauses of $\mathit{Unsat}(\mathcal{F}, \tau)$ is necessary for $\mathcal{F}$. Note: In particular, when $\mathit{Unsat}(\mathcal{F}, \tau) = \{C\}$ is a singleton – all variables of C are necessary.

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- So, flip v, but collect variables shared falsified clauses. Continue recursively.

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Another idea: translate to group-MUS.

- Doesn't work well, compared to the dedicated algorithm.

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- Construction of concise CNF encodings

Is Redundancy Common ?

A sample of 300 benchmarks from various applications of SAT

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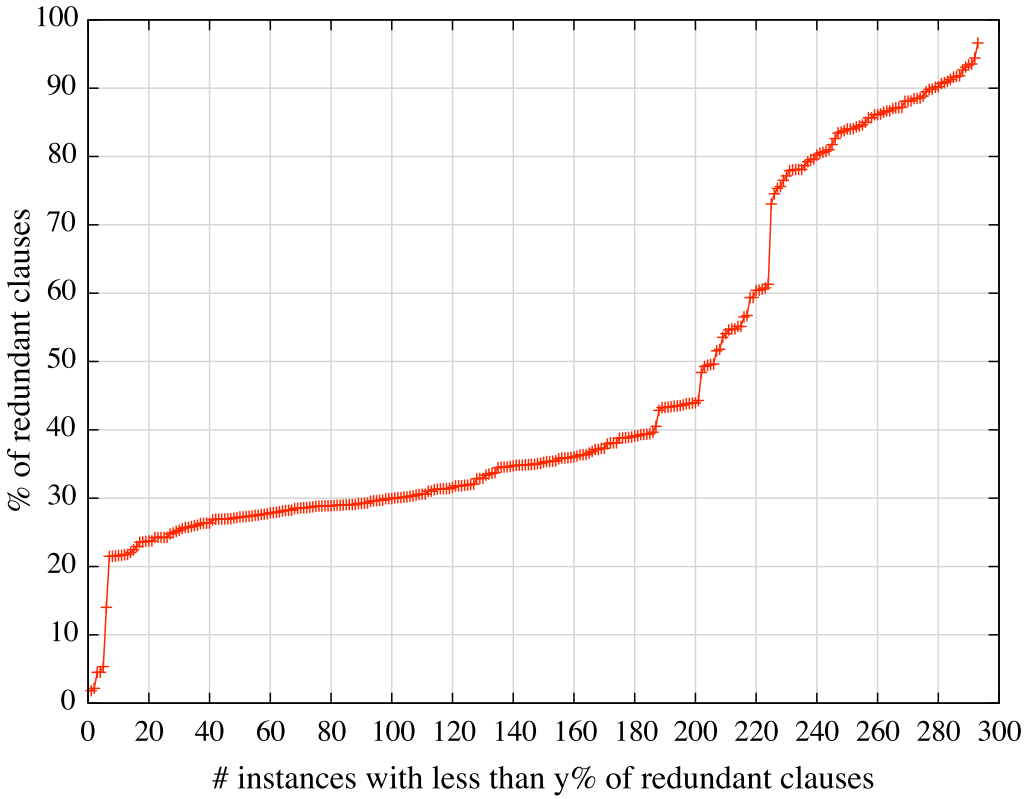

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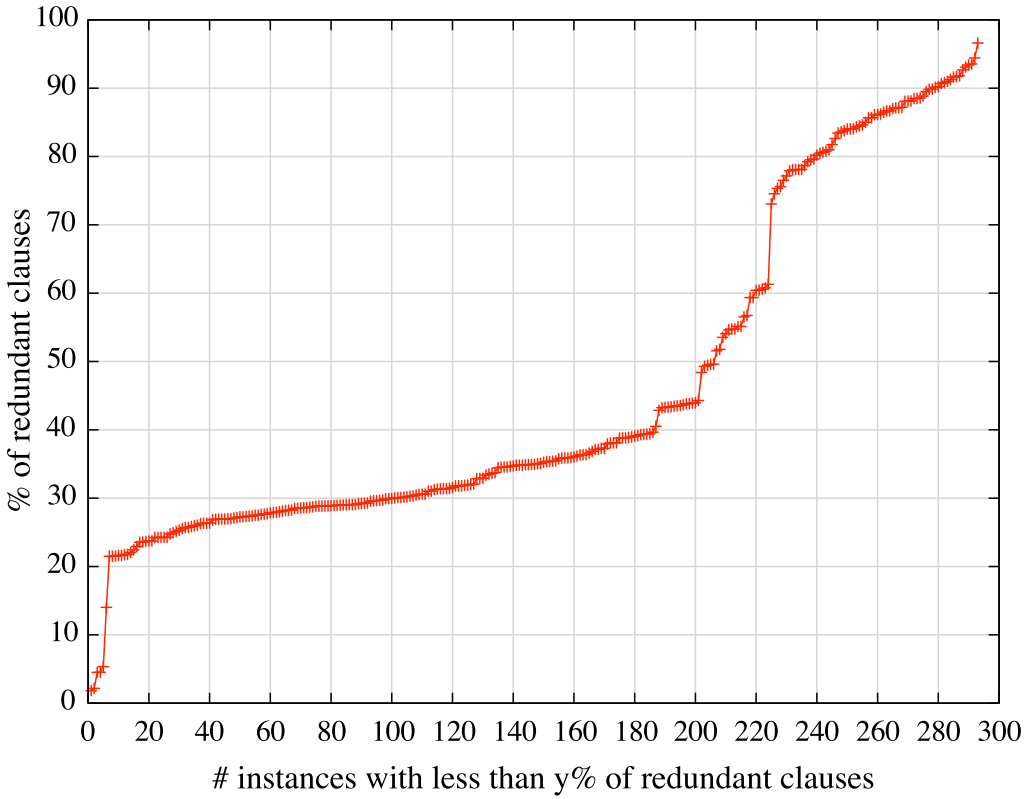

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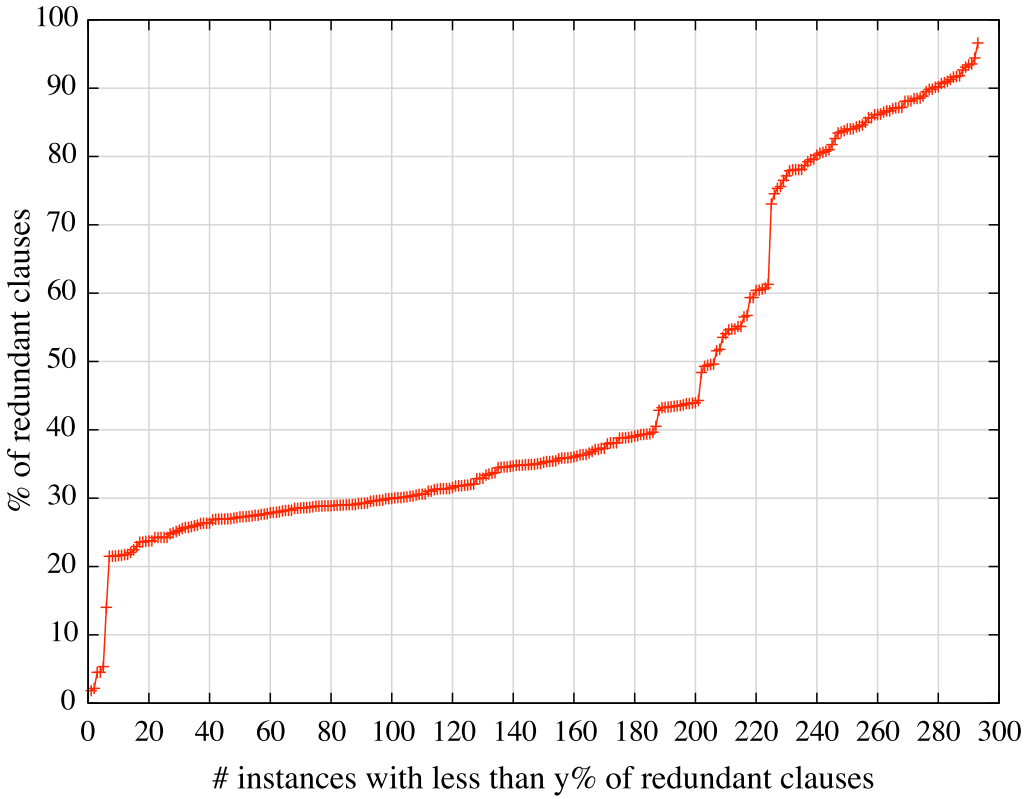

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- 1/3 has $>$ 50%, in some cases $>$ 90%, of redundant clauses !

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Deletion-based MES Computation

Input $\mapsto$ **Output**: CNF Formula $\mathcal{F}$ $\mapsto$ an MES $\mathcal{M}$ of $\mathcal{F}$

$\mathcal{M} \leftarrow \mathcal{F}$ // Inv: $\mathcal{M}$ is a superset of some MES of $\mathcal{F}$
foreach $C \in \mathcal{M}$ **do**
  if not SAT($\mathcal{M} \setminus \{C\} \cup \mathsf{CNF}(\neg C)$) **then** // is C redundant ?
    // yes - delete it
    $\mathcal{M} \leftarrow \mathcal{M} \setminus \{C\}$
  // no - keep it
return $\mathcal{M}$ // Every $C \in \mathcal{M}$ is irredundant in $\mathcal{M}$

Computing MESes [Belov et al, CP'12]

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- $\mathcal{F}$ – the input formula,
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- SAT check needs to be modified: e.g. the algorithm loops while SAT($\mathcal{M} \cup \mathcal{S} \cup$ CNF($\neg \mathcal{F}$) Note: in fact, do not need to negate the whole $\mathcal{F}$.

Computing MESes: optimizations

Model rotation:

- Fact: C is irredundant in $\mathcal{F}$ **iff** for τ, $\mathit{Unsat}(\mathcal{F}, \tau) = \{C\}$. Note: the same property as in MUSes.

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Clause-set refinement cannot be applied soundly to MES extraction !

Example: check C_2 in $\mathcal{F} = \{C_1 = (p), C_2 = (p \vee q), C_3 = (r)\}$.

Computing MESes: reduction to group-MUS

Observation

- $\mathcal{F}' \subseteq \mathcal{F}$ is an MES of $\mathcal{F}$ **iff**:
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- This is an instance of *group-MUS* computation problem !
  - take "group 0" to be $\text{CNF}(\neg\mathcal{F})$, and
  - make a singleton group for each $C \in \mathcal{F}$

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- Clause-set refinement is sound and effective again.
- Turns out that reduction can be done incrementally: family of algorithms (that includes the deletion-based algorithm).

MUSer2 — MUS, group-MUS, VMUS, MES extractor

Algorithms:

- Hybrid algorithm
- Insertion-based
- Dichotomic
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SAT solvers are used in a black-box manner; can use various SAT solvers.

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Thank you for your attention !