Foundations of Data and Knowledge Systems

EPCL Basic Training Camp 2012

Part Four

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December 20, 2012

Outline

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Minimal Model Semantics of Definite Rules

Recall

- Definite programs are finite sets of definite clauses, also called definite rules: $A \leftarrow B_1 \wedge \ldots \wedge B_n$ with $n \geq 0$.
- Definite programs admit a very natural semantics definition:
  - Each program Π is satisfiable.
  - The intersection of all its Herbrand models is a model of Π.
  - This is the *minimal model* of Π.
  - Precisely the atoms implied by Π are true in the minimal model.

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  - The intersection of all its Herbrand models is a model of Π.
  - This is the *minimal model* of Π.
  - Precisely the atoms implied by Π are true in the minimal model.
- Definite rules are a special case of universal and inductive formulas.
- The interesting model-theoretic properties of definite rules are inherited from these more general classes of formulas.

Minimal Model Semantics of Definite Rules ctd.

Recall

- A formula is universal, if it can be transformed into a prenex form with universal quantifiers only.
- A formula is inductive, if it can be transformed into a prenex form with the following properties:
  - The quantifier prefix starts with universal quantifiers for all variables in the consequent followed by arbitrary quantifiers for the remaining variables.
  - The quantifier-free part is of the form $(A_1 \wedge \ldots \wedge A_n) \leftarrow \varphi$, where $n \geq 0$ and φ is a *positive* formula (i.e., it contains no negation).
- An inductive formula is either a generalised definite rule (if $n \geq 1$) or a generalised definite goal (if $n = 0$).

Theorem

Each set S of definite rules (i.e., each definite program) has a unique minimal Herbrand model. This model is the intersection of all Herbrand models of S. It satisfies precisely those ground atoms that are logical consequences of S.

Outline

Minimal Model Construction

Outline

- The minimal models semantics is not constructive.
- We need algorithms to compute the / reason from the minimal model
- Different methods exist, including
  - algebraic approaches (fixpoints of consequence operators, “bottom up”)
  - proof-theoretic approaches (special resolution procedures, “top down”)
- We consider here first fix-point construction, for which we need concepts from operator theory.
- We confine here to a specific case of operators, applied to elements M of the powerset $\mathcal{P}(X)$ (the set of subsets) of a set X.

Operators

Definition (Operator)

Let X be a set. An operator on X is a mapping $\Gamma : \mathcal{P}(X) \to \mathcal{P}(X)$.

Definition (Monotonic operator)

Let X be a set. An operator Γ on X is *monotonic*, iff for all subset $M \subseteq M' \subseteq X$ it holds that: $\Gamma(M) \subseteq \Gamma(M')$.

Definition (Continuous operator)

Let X be a nonempty set.

A set $Y \subseteq \mathcal{P}(X)$ of subsets of X is *directed*, if every finite subset of Y has an *upper bound* in Y, i.e., for each finite $Y_{fin} \subseteq Y$, there is a set $M \in Y$ such that $\bigcup Y_{fin} \subseteq M$.

An operator Γ on X is *continuous*, iff for each directed set $Y \subseteq \mathcal{P}(X)$ of subsets of X it holds that: $\Gamma(\bigcup Y) = \bigcup\{\Gamma(M) \mid M \in Y\}$.

Continuous vs Monotone Operators

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Proof.

Let Γ be a continuous operator on $X \neq \emptyset$. Let $M \subseteq M' \subseteq X$. Since Γ is continuous, $\Gamma(M') = \Gamma(M \cup M') = \Gamma(M) \cup \Gamma(M')$, thus $\Gamma(M) \subseteq \Gamma(M')$. □

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The converse does not hold.

Example

Let $\Gamma(X) = \emptyset$, if X is finite, and $\Gamma(X) = X$, if X is infinite.

- Γ is monotonic.
- Γ is not continuous in general. E.g., let $X = \mathbb{N}$ and $Y = \{\{0, 1, \ldots n\} \mid n \in \mathbb{N}\}$.
  Then $\Gamma(\bigcup Y) = \mathbb{N}$ but $\bigcup_{M \in Y} \Gamma(M) = \emptyset$.

Fixpoints of Monotonic and Continuous Operators

Definition (Fixpoint)

Let Γ be an operator on a set X. A subset $M \subseteq X$ is

- a *pre-fixpoint* of Γ iff $\Gamma(M) \subseteq M$;
- a *fixpoint* of Γ iff $\Gamma(M) = M$.

Theorem (Knaster-Tarski, existence of least and greatest fixpoint)

Let Γ be a monotonic operator on a nonempty set X. Then Γ has a least fixpoint $lfp(\Gamma)$ and a greatest fixpoint $gfp(\Gamma)$ with

$$\begin{aligned} lfp(\Gamma) &= \bigcap\{M \subseteq X \mid \Gamma(M) = M\} = \bigcap\{M \subseteq X \mid \Gamma(M) \subseteq M\}. \\ gfp(\Gamma) &= \bigcup\{M \subseteq X \mid \Gamma(M) = M\} = \bigcup\{M \subseteq X \mid \Gamma(M) \subseteq M\}. \end{aligned}$$

- This is a fundamental result with many applications in Computer Science.
- It holds for more general structures (complete partial orders).

Proof.

For the least fixpoint let $L = \bigcap\{M \subseteq X \mid \Gamma(M) \subseteq M\}$.

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Consider an arbitrary $M \subseteq X$ with $\Gamma(M) \subseteq M$.

By definition of L we have $L \subseteq M$. Since Γ is monotonic, $\Gamma(L) \subseteq \Gamma(M)$. With the assumption $\Gamma(M) \subseteq M$ follows $\Gamma(L) \subseteq M$. Therefore

$$\Gamma(L) \subseteq \bigcap\{M \subseteq X \mid \Gamma(M) \subseteq M\} = L. \tag{1}$$

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$$\Gamma(L) \subseteq \bigcap\{M \subseteq X \mid \Gamma(M) \subseteq M\} = L. \quad (1)$$

For the opposite inclusion, from (1) and since Γ is monotonic it follows that $\Gamma(\Gamma(L)) \subseteq \Gamma(L)$. By definition of L, $\Gamma(L) \in \bigcap\{M \subseteq X \mid \Gamma(M) \subseteq M\}$; therefore

$$L \subseteq \Gamma(L). \quad (2)$$

From (1) and (2) it follows that L is a fixpoint of Γ.

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Now let $L' = \bigcap\{M \subseteq X \mid \Gamma(M) = M\}$.

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The proof for the greatest fixpoint is similar. □

Ordinal Numbers and Powers

- Ordinal numbers are the *order types* of *well-ordered sets* (i.e., totally ordered sets where each set has a minimum.)
- They generalize natural numbers, and can be defined as well-ordered sets of all smaller ordinals, or as *hereditarily transitive sets* A: (i) If $x \in A$ and $y \in x$, then $y \in A$; (ii) each $x \in A$ is transitive.
- They are divided into *successor ordinals* β, given by $\beta = \alpha + 1$ for ordinal α, and limit ordinals λ (not of this form).
- The first limit ordinal, ω, corresponds to the set $\mathbb{N}$ of all natural numbers.

Definition (Ordinal powers of a monotonic operator)

Let Γ be a monotonic operator on a nonempty set X. For each ordinal β, the *upward and downward power* of Γ, $\Gamma \uparrow \beta$ and $\Gamma \downarrow \beta$ is defined as

$\Gamma \uparrow 0 = \emptyset$	$\beta = 0$ (base)	$\Gamma \downarrow 0 = X$
$\Gamma \uparrow \alpha{+}1 = \Gamma(\Gamma \uparrow \alpha)$	$\beta = \alpha + 1$ (succ.)	$\Gamma \downarrow \alpha{+}1 = \Gamma(\Gamma \downarrow \alpha)$
$\Gamma \uparrow \lambda = \bigcup\{\Gamma \uparrow \beta \mid \beta < \lambda\}$	$\beta = \lambda$ (limit)	$\Gamma \downarrow \lambda = \bigcap\{\Gamma \downarrow \beta \mid \beta < \lambda\}$

Lemma

Let Γ be a monotonic operator on a nonempty set X. For each ordinal α holds:

1. $\Gamma \uparrow \alpha \quad \subseteq \quad \Gamma \uparrow \alpha + 1$
2. $\Gamma \uparrow \alpha \quad \subseteq \quad \mathit{lfp}(\Gamma)$.
3. *If* $\Gamma \uparrow \alpha = \Gamma \uparrow \alpha + 1$, *then* $\mathit{lfp}(\Gamma) = \Gamma \uparrow \alpha$.

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Proof (Idea).

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Item 3.: If $\Gamma \uparrow \alpha = \Gamma \uparrow \alpha + 1$, then $\Gamma \uparrow \alpha = \Gamma(\Gamma \uparrow \alpha)$, i.e., $\Gamma \uparrow \alpha$ is a fixpoint of Γ, therefore $\Gamma \uparrow \alpha \subseteq \mathit{lfp}(\Gamma)$ by 2., and $\mathit{lfp}(\Gamma) \subseteq \Gamma \uparrow \alpha$ by definition. □

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Theorem

For any monotonic operator Γ on $X \neq \emptyset$, $\mathit{lfp}(\Gamma) = \Gamma \uparrow \alpha$ for some ordinal α.

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Proof.

If not, for all ordinals α by the previous lemma $\Gamma \uparrow \alpha \subseteq \Gamma \uparrow \alpha + 1$ and $\Gamma \uparrow \alpha \neq \Gamma \uparrow \alpha + 1$. Thus $\Gamma \uparrow$ maps the ordinals 1-1 to (a subset of) $\mathcal{P}(X)$, a contradiction (there are “more” ordinals than any set can have elements). □

Least Fixpoint of Continuous Operator

Theorem (Kleene)

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Proof.

By 1. from the previous lemma, it suffices to show that $\Gamma \uparrow \omega + 1 = \Gamma \uparrow \omega$.

$$\begin{aligned}
\Gamma \uparrow \omega + 1 &= \Gamma(\Gamma \uparrow \omega) && \text{by definition, successor case}\\
&= \Gamma\big(\bigcup\{\Gamma \uparrow n \mid n \in \mathbb{N}\}\big) && \text{by definition, limit case}\\
&= \bigcup\big\{\Gamma(\Gamma \uparrow n) \mid n \in \mathbb{N}\big\} && \text{because } \Gamma \text{ is continuous}\\
&= \bigcup\big\{\Gamma \uparrow n + 1 \mid n \in \mathbb{N}\big\} && \text{by definition, successor case}\\
&= \Gamma \uparrow \omega && \text{by definition, base case}
\end{aligned}$$

□

Note: An analogous result for the greatest fixpoint does not hold.

Outline

Immediate Consequence Operator

- We now apply the above results for universal generalized definite rules, i.e., of form $\forall^*((A_1 \wedge \cdots \wedge A_n) \leftarrow \varphi)$, where each A_i is an atom and φ is a quantifier-free *positive* formula.
- Here $X = HB$ and a subset M is a set $B \subseteq HB$ of ground atoms.

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Definition (Immediate consequence operator)

Let S be a set of universal generalised definite rules. Let $B \subseteq HB$ be a set of ground atoms. The *immediate consequence operator* $\mathbf{T}_S$ *for* S is:

$$\begin{aligned} \mathbf{T}_S : \mathcal{P}(HB) &\to \mathcal{P}(HB) \\ B &\mapsto \{A \in HB \mid \text{there is a ground instance } ((A_1 \wedge \ldots \wedge A_n) \leftarrow \varphi) \\ &\qquad \text{of a member of } S \text{ with } HI(B) \models \varphi \text{ and } A = A_i \\ &\qquad \text{for some } i \text{ with } 1 \leq i \leq n \} \end{aligned}$$

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Lemma ($\mathbf{T}_S$ is continuous)

Let S be a set of universal generalised definite rules. The immediate consequence operator $\mathbf{T}_S$ is continuous (hence, also monotonic).

Theorem

Let S be a set of universal generalised definite rules. Let $B \subseteq HB$ be a set of ground atoms. Then $HI(B) \models S$ *iff* $\mathbf{T}_S(B) \subseteq B$.

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"only if:" Assume $HI(B) \models S$. Let $A \in \mathbf{T}_S(B)$, i.e., $A = A_i$ for some ground instance $((A_1 \land \ldots \land A_n) \leftarrow \varphi)$ of a member of S with $HI(B) \models \varphi$.

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- If $HI(B) \not\models \varphi$, it does.
- If $HI(B) \models \varphi$, then $A_1 \in \mathbf{T}_S(B), \ldots, A_n \in \mathbf{T}_S(B)$ by definition of $\mathbf{T}_S$.

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  By assumption $A_1 \in B, \ldots, A_n \in B$.

  As all A_i are ground atoms, $HI(B) \models A_1, \ldots, HI(B) \models A_n$. Thus $HI(B)$ satisfies r.

□

Corollary (Fixpoint Characterization of the Least Herbrand Model)

Let S be a set of universal generalised definite rules. Then

(i) $\mathit{lfp}(\mathbf{T}_S) = \mathbf{T}_S \uparrow \omega = \mathit{Mod}_\cap(S) = \{A \in \mathit{HB} \mid S \models A\}$ *and*

(ii) $\mathit{HI}(\mathit{lfp}(\mathbf{T}_S))$ *is the unique minimal Herbrand model of S.*

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(ii) $HI(lfp(\mathbf{T}_S))$ *is the unique minimal Herbrand model of* S.

Proof.

(i): By the Lemma above, $\mathbf{T}_S$ is a continuous operator on HB, and by Kleene's Theorem, $lfp(\mathbf{T}_S) = \mathbf{T}_S \uparrow \omega$. Note that $Mod_{HB}(S) \neq \emptyset$ (as $HI(HB) \models S$)

Now,

$lfp(\mathbf{T}_S)$	$= \bigcap\{B \subseteq HB \mid \mathbf{T}_S(B) \subseteq B\}$	by the Knaster-Tarski Theorem
	$= \bigcap\{B \subseteq HB \mid HI(B) \models S\}$	by the previous Theorem
	$= \bigcap Mod_{HB}(S)$	by definition of Mod_{HB}
	$= Mod_\cap(S)$	by definition of $Mod_\cap$
	$= \{A \in HB \mid S \models A\}$	as S is universal (see unit 4)

(ii): By (i), $HI(lfp(\mathbf{T}_S))$ is the intersection of all Herbrand models of S, and $HI(lfp(\mathbf{T}_S)) \models S$, as S is satisfiable.

Hence, $HI(lfp(\mathbf{T}_S))$ is the unique minimal Herbrand model of S. □

Characterization Summary

- The "natural meaning" of a set S of universal generalised definite rules can defined in different but equivalent ways:
  - as the unique minimal Herbrand model of S;
  - as the intersection $HI(Mod_{\cap}(S))$ of all Herbrand models of S;
  - as the set $\{A \in HB \mid S \models A\}$ of ground atoms entailed by S;
  - as the least fixpoint $lfp(\mathbf{T}_S)$ of the immediate consequence operator
- Declarative and procedural (forward chaining) semantics coincide.
- Further equivalent procedural semantics, based on SLD resolution, exists (backward chaining).

Outline

Declarative Semantics of Rules with Negation

If a database of students does not list “Mary”, then one may conclude that “Mary” is not a student. The principle underlying this is called closed world assumption (CWA).

Two approaches to coping with this form of negation:

- axiomatization within first-oder predicate logic
- deduction methods not requiring specific axioms conveying the CWA

The second approach is desirable but it poses the problem of the declarative semantics, or model theory.

Not all Minimal Models convey the CWA

Example

- $S_1 = \{\, (q \leftarrow r \wedge \neg p),\ (r \leftarrow s \wedge \neg t),\ (s \leftarrow \top) \,\}$
  Minimal Herbrand models: $HI(\{s, r, q\})$, $HI(\{s, r, p\})$, and $HI(\{s, t\})$.
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- $S_4 = \{\, (p \leftarrow \neg p),\ (p \leftarrow \top)\,\}$
  Minimal Herbrand model: $HI(\{p\})$.
  Here, p is arguably justified and S_4 should be consistent.

  Note: different from classical logic, a subset of a consistent rule set ($S_3 \subseteq S_4$) may be inconsistent!

Justification and Consistency Postulate

Summarizing the above examples:

Justification Postulate

Derived truths must have “justifications” in terms of rules.

- In S_1 above, only in $HI(\{s, r, q\})$ all atoms have justifications.
- The only rule of S_3 does not “justify” p

Consistency Postulate

Every syntactically correct set of normal clauses is consistent (as it has a classical model) and must therefore have a model.

- S_3 must have a model, the only Herbrand candidate is $HI(\{p\})$.

Non-Monotonic Consequence

- A consequence operator is a mapping that assigns a set S of formulas a set of formulas $Th(S)$ (satisfying certain properties).
- We can view $Th(S)$ as an operator considered above.
- S_3 and S_4 suggest that a consequence operator for rules with negation should be non-monotonic (if $Th(S)$ for “inconsistent” S yields all formulas).

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$S_5 = \{\, (q \leftarrow \neg p) \,\}$ has the minimal Herbrand models: $HI(\{p\})$ and $HI(\{q\})$.
Only $HI(\{q\})$ conveys the intuitive meaning under the CWA and should be retained as (the only) canonical model. Therefore, $q \in Th_{can}(S_5)$.

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$S_5' = S_5 \cup \{\, (p \leftarrow \top) \,\}$ has the single minimal Herbrand model $HI(\{p\})$, which also conveys the intuitive meaning under the CWA and should be retained as a canonical model. Therefore, $q \notin Th_{can}(S_5')$.

Thus, $S_5 \subseteq S_5'$, but $Th_{can}(S_5) \not\subseteq Th_{can}(S_5')$.

Outline

Stratifiable Rule Sets

Basic Idea

Avoid cases like $(p \leftarrow \neg p)$ and more generally recursion through negative literals.

Definition (Stratification)

A stratification of a set S of normal clauses (rules) is a partition $S_0, \ldots, S_k$ of S such that

- For each relation symbol p there is a stratum S_i, such that all clauses of S containing p in their consequent are members of S_i.
  In this case one says that the relation symbol p is defined in stratum S_i.
- For each stratum S_j and positive literal A in the antecedents of members of S_j, the relation symbol of A is defined in a stratum S_i with $i \leq j$.
- For each stratum S_j and negative literal $\neg A$ in the antecedents of members of S_j, the relation symbol of A is defined in a stratum S_i with $i < j$.

A set of normal clauses is called stratifiable, if there exists a stratification of it.

Example

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- The set $S = \{\, (r \leftarrow \top),\ (q \leftarrow r),\ (p \leftarrow q \wedge \neg r) \,\}$ is stratifiable: the stratum S_0 contains the first clause and the stratum S_1 the last one, while the middle clause may belong to either of the strata.

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- The set $S = \{ (p \leftarrow \neg p) \}$ is not stratifiable.
- Any set of normal clauses with a “cycle of recursion through negation” (defined syntactically via a *dependency graph* is not stratifiable.

Stratifiable Rule Sets – Canoncial Model

Principal Idea

- The stratum S_0 always consists of definite clauses (positive definite rules).
- Hence the truth values of all atoms of stratum S_0 can be determined without negation being involved.

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That is, *work stratum by stratum*.

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Stratification Theorem (Apt, Blair and Walker)

Each stratifiable rule set has a well-defined canonical model (also called *perfect model*), which is *independent of a particular stratification*.

Outline

Stable Model Semantics

Basic Idea

Perform *assumption-based* evaluation, where negation takes the value in the final result.

Definition (Gelfond-Lifschitz transformation)

Let S be a (possibly infinite) set of *ground* normal clauses, i.e., of formulas

$$A \leftarrow L_1 \wedge \ldots \wedge L_n$$

where $n \geq 0$ and A is a ground atom and the L_i for $1 \leq i \leq n$ are ground literals. Let $B \subseteq HB$. The Gelfond-Lifschitz transform $GL_B(S)$ of S with respect to B is obtained from S as follows:

1. remove each clause whose antecedent contains a literal $\neg A$ with $A \in B$.
2. remove from the antecedents of the remaining clauses all negative literals.

Definition (Stable model)

Let S be a (possibly infinite) set of ground normal clauses. An Herbrand interpretation $HI(B)$ is a *stable model of S* iff it is the unique minimal Herbrand model of $GL_B(S)$.
A stable model of a set S of normal clauses is a stable model of the (possibly infinite) set of ground instances of S.

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Stable Model Semantics –Properties

Theorem

Every stable model of a normal clause set S is a minimal Herbrand model of S.

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Proof.

It suffices to consider a set S of *ground* normal clauses.

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As easily seen from the definition of $GL_B(\cdot)$, this implies $HI(B) \models S$.

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Let $C \in GL_B(S)$. Then C results from some clause $D \in S$, by removing the negative literals from its antecedent. If $\neg A$ is such a literal, then $A \notin B$, and, since $B' \subseteq B$, also $A \notin B'$. Therefore, $C \in GL_{B'}(S)$. As $HI(B') \models S$, it follows $HI(B') \models C$. □

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Proposition

Every stratifiable rule set has exactly one stable model, which coincides with the respective canonical model.

Stable Model Semantics – Evaluation

- The Stable Model Semantics coincides with the intuitive understanding based on the Justification Postulate.
- It does not satisfy the Consistency Postulate.
- It gracefully generalizes the canonical semantics.
- To date, Stable Model Semantics is the predominant multiple model non-montonic semantics for rule sets with negation.

Appendix: Beyond Herbrand Models

Generalisation

- Minimal Models are also defined for non-Herbrand interpretations
- They make sense also for generalizations of non-inductive formulas
- Uniqueness and intersection property might be lost
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A generalised rule is a formula of the form $\forall^*(\psi \leftarrow \varphi)$ where φ is positive and ψ is positive and quantifier-free.

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Generalised rules are not necessarily universal: $p(a) \leftarrow \forall x.q(x)$

Supportedness in Minimal Models

Definition (Supported Atoms)

Let $\mathcal{I}$ be an interpretation, V a variable assignment in $dom(\mathcal{I})$ and $A = p(t_1, \ldots, t_n)$ an atom, $n \geq 0$.

- an atom B supports A in $\mathcal{I}[V]$ iff $\mathcal{I}[V] \models B$ and $B = p(s_1, \ldots, s_n)$ and $s_i^{\mathcal{I}[V]} = t_i^{\mathcal{I}[V]}$ for $1 \leq i \leq n$.
- a set C of atoms supports A in $\mathcal{I}[V]$ iff $\mathcal{I}[V] \models C$ and there is an atom in C that supports A in $\mathcal{I}[V]$.
- a generalised rule $\forall^*(\psi \leftarrow \varphi)$ supports A in I iff for each variable assignment V with $\mathcal{I}[V] \models \varphi$ there exists an implicant C of ψ that supports A in $\mathcal{I}[V]$.

Informally, an implicant C of ψ is a set of atoms which logically implies ψ

Implicant of a Positive Quantifier-Free Formula

Definition (Pre-Implicant and Implicant)

Let ψ be a positive quantifier-free formula. The set $primps(\psi)$ of pre-implicants of ψ is defined as follows:

- $primps(\psi) = \{ \{\psi\} \}$ if ψ is an atom or $\top$ or $\bot$.
- $primps(\neg\psi_1) = primps(\psi_1)$.
- $primps(\psi_1 \wedge \psi_2) = \{ C_1 \cup C_2 \mid C_1 \in primps(\psi_1),\ C_2 \in primps(\psi_2) \}$.
- $primps(\psi_1 \vee \psi_2) = primps(\psi_1 \Rightarrow \psi_2) = primps(\psi_1) \cup primps(\psi_2)$.

The set of implicants of ψ is obtained from $primps(\psi)$ by removing all sets containing $\bot$ and by removing $\top$ from the remaining sets.

Lemma

1. *If C is an implicant of ψ, then $C \models \psi$.*
2. *For any interpretation $\mathcal{I}$ and variable assignment V in $dom(\mathcal{I})$, if $\mathcal{I}[V] \models \psi$ then there exists an implicant C of ψ with $\mathcal{I}[V] \models C$.*

Supportedness by Generalized Rules, Revisited

Reconsider the definition:

Let $\mathcal{I}$ be an interpretation and $A = p(t_1, \ldots, t_n)$ an atom, $n \geq 0$. Then

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Fix: allow variables “outside” A to change value.

Supportedness Result

Theorem (Minimal Models Satisfy Only Supported Ground Atom)

Let S be a set of generalised rules. If $\mathcal{I}$ is a minimal model of S, then for each ground atom A with $\mathcal{I} \models A$ there exists some generalised rule in S that supports A in $\mathcal{I}$.

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Example

Consider a signature with a unary relation symbol p and constants a and b.
Let $S = \{\ (p(b) \leftarrow \top)\ \}$.
The interpretation $\mathcal{I}$ with $dom(\mathcal{I}) = \{1\}$ and $a^{\mathcal{I}} = b^{\mathcal{I}} = 1$ and $p^{\mathcal{I}} = \{(1)\}$ is a minimal model of S.
Moreover, $\mathcal{I} \models p(a)$. By the theorem, $p(a)$ is supported in $\mathcal{I}$ by $p(b)$, which can be confirmed by applying the definition.

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Non-Minimal Supportedness

The converse of the Theorem fails, e.g. $S = \{ (p \leftarrow p) \}$.

Proof.

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Let $\mathcal{I}'$ be identical to $\mathcal{I}$ except that $\mathcal{I}' \not\models A$. Then $\mathcal{I}' < \mathcal{I}$.

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  By part 1 of the above Lemma , $\mathcal{I}'[V] \models \psi$. Hence $\mathcal{I}'[V] \models (\psi \leftarrow \varphi)$.

In all possible cases $\mathcal{I}'$ satisfies r; thus $\mathcal{I}'$ is a model of S, contradicting the minimality of $\mathcal{I}$. □

Semantic vs Syntactic Support

- The above theorem is semantic in nature:
  In the above example, $p(a)$ is supported by $p(b)$
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Definition (Unique Name Property)

An interpretation $\mathcal{I}$ has the *unique name property*, if for each term s, ground term t, and variable assignment V in $dom(\mathcal{I})$ with $s^{\mathcal{I}[V]} = t^{\mathcal{I}[V]}$ there exists a substitution σ with $s\sigma = t$.

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- Herbrand interpretations have the unique name property.
- The relationship between the supporting atom and the supported ground atom specialises to the (syntactic and decidable) ground instance relationship.
- Sometimes, unique names are postulated (*Unique Names Assumption*)

Well-Founded Semantics

Basic Idea

- Avoid cases like $(p \leftarrow \neg p)$ by using a third truth value, *unkown*.
- Try to build a single *partial* model, in which p would be unknown.

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Notation

For a literal L, $\overline{L}$ is its complement with $\overline{A} = \neg A$ and $\overline{\neg A} = A$ for an atom A. For a set I of ground literals, $\overline{I} = \{\, \overline{L} \mid L \in I \,\}$, $\quad pos(I) = I \cap HB, \quad neg(I) = \overline{I} \cap HB$. Thus, $I = pos(I) \cup \overline{neg(I)}$.

Definition

- A set I of ground literals is *consistent*, if $pos(I) \cap neg(I) = \emptyset$, else *inconsistent*.
- Sets I_1, I_2 of ground literals are *(in)consistent* if $I_1 \cup I_2$ is (in)consistent.
- A ground literal L and a set I of ground literals are *(in)consistent* if $\{L\} \cup I$ is (in)consistent.

Definition (Partial interpretation)

A *partial interpretation* is a consistent set I of ground literals; it is *total*, iff $pos(I) \cup neg(I) = HB$, i.e., for each ground atom A either $A \in I$ or $\neg A \in I$.
For a total I, the Herbrand interpretation induced by I is $HI(I) = HI(pos(I))$.

Definition (Satisfaction for partial interpretations)

Let I be a partial interpretation.

Then $\top$ is *satisfied* in I and $\bot$ is *falsified* in I.

A ground literal L is

satisfied or true in I	iff	$L \in I$.
falsified or false in I	iff	$\overline{L} \in I$.
undefined in I	iff	$L \notin I$ and $\overline{L} \notin I$.

A conjunction $L_1 \wedge \ldots \wedge L_n$ of ground literals, $n \geq 0$, is

satisfied or true in I	iff	each L_i for $1 \leq i \leq n$ is satisfied in I.
falsified or false in I	iff	at least one L_i for $1 \leq i \leq n$ is falsified in I.
undefined in I	iff	each L_i for $1 \leq i \leq n$ is satisfied or undefined in I and at least one of them is undefined in I.

Definition (Satisfaction, ctd)

Let I be a partial interpretation.
A ground normal clause $A \leftarrow \varphi$ is

satisfied or true in I	iff	A is satisfied in I or φ is falsified in I.
falsified or false in I	iff	A is falsified in I and φ is satisfied in I.
weakly falsified in I	iff	A is falsified in I and φ is satisfied or undefined in I.

A normal clause is

satisfied or true in I	iff	each of its ground instances is.
falsified or false in I	iff	at least one of its ground instances is.
weakly falsified in I	iff	at least one of its ground instances is.

A set of normal clauses is

satisfied or true in I	iff	each of its members is.
falsified or false in I	iff	at least one of its members is.
weakly falsified in I	iff	at least one of its members is.

- Note: "weakly falsified" intuitively means that by turning from "undefined" to "true", the clause could be falsified.
- For a total interpretation I, the cases "undefined" and "weakly falsified" are impossible, and satisfaction in $HI(I)$ amounts to the classical notion.

Definition (Total and partial model)

Let S be a set of normal clauses.

- A total interpretation I is a total model of S, iff S is satisfied in I.
- A partial interpretation I is a partial model of S, iff there exists a total model I' of S with $I \subseteq I'$.

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Lemma (Weak Falsification)

Let S be a set of normal clauses and I a partial interpretation. If no clause in S is weakly falsified in I, then I is a partial model of S.

Unfounded Sets

Principle for Drawing Negative Conclusions

Given a partial interpretation I, a set U of ground atoms is "unfounded" wrt a clause set, if each atom A in U is unjustified wrt I, *taking U into account*.

Example

Let $S = \{(p \leftarrow q),\ (q \leftarrow p)\}$. For $U = \{p, q\}$, p, q are unjustified wrt $\{p, q\}$.

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Example

Let $S = \{(p \leftarrow q),\ (q \leftarrow p)\}$. For $U = \{p, q\}$, p, q are unjustified wrt $\{p, q\}$.

Definition (Unfounded set of ground atoms)

Let S be a set of normal clauses, and I a partial interpretation.
A set $U \subseteq HB$ of ground atoms is an unfounded set wrt S and I, if for each $A \in U$ and for each ground instance $r = A \leftarrow L_1 \wedge \ldots \wedge L_n$, $n \geq 1$, of a member of S, at least one of the following holds:

1. $L_i \in \overline{I}$ for some positive or negative L_i with $1 \leq i \leq n$. (L_i is falsified in I)
2. $L_i \in U$ for some positive L_i with $1 \leq i \leq n$. (L_i is unfounded)

A respective L_i is a *witness of unusability* for r.

Example

- Let $S = \{(p \leftarrow q),\ (q \leftarrow p)\}$.
  Then $U = \{p, q\}$ is an unfounded set wrt S and $I = \{p, q\}$.
  Both a and b are unfounded by condition 2.
- Let $S' = \{\ (q \leftarrow p),\ (r \leftarrow s),\ (s \leftarrow r)\ \}$ and $I = \{\neg p, \neg q\}$.
  The set $U' = \{q, r, s\}$ is unfounded wrt S' and I.
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  The atom q is unfounded by condition 1, the atoms r and s by condition 2.

Lemma

Let S be a set of normal clauses and I a partial interpretation. There exists a unique maximal (under set inclusion) unfounded set wrt. S and I, denoted $GUS_S(I)$,
Moreover, $GUS_S(I)$ is the union of all unfounded sets wrt. S and I.

Example (cont'd)

$GUS_S(I) = \{p, q\}$ and $GUS_{S'}(I') = \{p, q, r, s\}$

Observation

- If all atoms in I are founded, by switching any unfounded atom(s) to false (and further affected unfounded atoms), all rules remain satisfied.
- As no backtracking is needed, unfounded atoms can be safely made false.

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Lemma

Let S be a set of normal clauses, I be a partial interpretation, and U' be an unfounded set wrt. S and I, such that $pos(I) \cap U' = \emptyset$.
For each $U \subseteq U'$, its remainder $U' \setminus U$ is unfounded w.r.t. S and $I \cup \overline{U}$.

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A kind of opposite property is that false atoms are unfounded.

Lemma

Let S be a set of normal clauses and $I = pos(I) \cup \overline{neg(I)}$ be a partial interpretation. If no clause in S is weakly falsified in I (i.e., I is a partial model of S), then $neg(I)$ is unfounded wrt. S and $pos(I)$.

The above properties are exploited to extend a partial interpretation.

Definition (Operators $\mathbf{T}_S$, $\mathbf{U}_S$, $\mathbf{W}_S$)

Let $\mathcal{PI} = \{\, I \subseteq HB \cup \overline{HB} \mid I \text{ is consistent} \,\}$, and note that $\mathcal{P}(HB) \subseteq \mathcal{PI}$. Let S be a set of normal clauses. We define three operators:

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\begin{array}{llll}
\mathbf{T}_S: & \mathcal{PI} & \to & \mathcal{P}(HB) \\
 & I & \mapsto & \{\, A \in HB \mid \text{there is a ground instance } (A \leftarrow \varphi) \\
 & & & \quad \text{of a member of } S \text{ such that } \varphi \text{ is satisfied in } I \,\} \\
\mathbf{U}_S: & \mathcal{PI} & \to & \mathcal{P}(HB) \\
 & I & \mapsto & \text{the maximal subset of } HB \text{ that is unfounded wrt } S \text{ and } I \\
\mathbf{W}_S: & \mathcal{PI} & \to & \mathcal{PI} \\
 & I & \mapsto & \mathbf{T}_S(I) \cup \overline{\mathbf{U}_S(I)}
\end{array}
$$

- Starting from “knowing” I, the ground atoms in $\mathbf{T}_S(I)$ have to be true;
- those in $\mathbf{U}_S(I)$ are unfounded;
- $\mathbf{T}_S(I) \cap \mathbf{U}_S(I) = \emptyset$, thus $\mathbf{W}_S(I)$ is consistent.

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Lemma

$\mathbf{T}_S$, $\mathbf{U}_S$, *and* $\mathbf{W}_S$ *are monotonic.*

Example

Suppose $HB = \{p, q, r, s, t\}$, and let $I_0 = \emptyset$ and
$S = \{\, (q \leftarrow r \wedge \neg p),\ (r \leftarrow s \wedge \neg t),\ (s \leftarrow \top)\,\}$.

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\begin{aligned}
\mathbf{T}_S(I_0) &= \{s\} \\
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Theorem (Existence of least fixpoint)

Let S be a set of normal clauses. Then

(1) $\mathbf{W}_S$ *has a least fixpoint given by*

$$lfp(\mathbf{W}_S) = \bigcap\{I \in \mathcal{PI} \mid \mathbf{W}_S(I) = I\} = \bigcap\{I \in \mathcal{PI} \mid \mathbf{W}_S(I) \subseteq I\}.$$

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(3) $lfp(\mathbf{W}_S)$ *is a partial model of S.*

Theorem (Existence of least fixpoint)

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Proof.

(1): Knaster-Tarski Theorem. (2): show consistency and that no clause in S is weakly falsified by transfinite induction. (3): Weak Falsification Lemma. □

Definition (Well-founded model)

The well-founded model of a set S of normal clauses is $lfp(\mathbf{W}_S)$.

- The well-founded model may be total (it specifies a truth value for each ground atom) or partial (it leaves some atoms undefined).
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Example

- $S_1 = \{\, (q \leftarrow r \wedge \neg p),\ (r \leftarrow s \wedge \neg t),\ (s \leftarrow \top)\,\}$ has the well-founded model $\{s, r, q, \neg p, \neg t\}$. It is total.
- $S_2 = \{\, (p \leftarrow \neg q),\ (q \leftarrow \neg p)\,\}$ has the well-founded model $\emptyset$. It is partial and leaves the truth values of p and of q undefined.
- $S_3 = \{\, (p \leftarrow \neg p)\,\}$ has the well-founded model $\emptyset$. It is partial and leaves the truth value of p undefined.
- $S_4 = \{\, (p \leftarrow \neg p),\ (p \leftarrow \top)\,\}$ has the well-founded model $\{p\}$. It is total.

Well-Founded Semantics - Evaluation

- The well-founded semantics (WFS) coincides with an intuitive understanding based on the “Justification Postulate”.
- A set of normal clauses always has exactly one well-founded model, but some ground atoms might be “undefined” in it (they can be defined, however). Thus, WFS coincides with the “Consistency Postulate”.
- The well-founded model might not be computable (in those not infrequent cases where the fixpoint is reached after more than ω steps).

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$S = \{ p(a) \leftarrow \top, \quad p(f(x)) \leftarrow p(x), \quad q(y) \leftarrow p(y), \quad s \leftarrow p(z) \wedge \neg q(z), \quad r \leftarrow \neg s \}$

is the (standard) translation of the following set of generalised rules

$$\{ p(a) \leftarrow \top, \qquad p(f(x)) \leftarrow p(x), \qquad q(y) \leftarrow p(y), \qquad r \leftarrow \forall z\big(p(z) \Rightarrow q(z)\big) \}$$

into normal clauses. Then

$$\begin{aligned} lfp(\mathbf{W}_S) &= \mathbf{W}_S \uparrow \omega + 2 \\ &= \{ p(a), \ldots, p(f^n(a)), \ldots \} \cup \{ q(a), \ldots, q(f^n(a)), \ldots \} \cup \{ \neg s, \ r \} \end{aligned}$$

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- Stable model entailment does *not* imply well-founded entailment:

Example

Let $S = \{p \leftarrow \neg q,\;\; q \leftarrow \neg p,\;\; r \leftarrow p,\;\; r \leftarrow q\}$.
Then r is true in all stable models but it is undefined in the well-founded model.

"reasoning by cases"